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# Finite Groups with Regular Orbits on Vector Spaces

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In this article we investigate the following situation, which is interesting, for example, in the theory of character degrees:

Let $A$ be a finite group, $\mathbb{F}_q$ the finite field with $q = p^s$ elements, and $V$ a finite-dimensional $\mathbb{F}_q A$-module. During the whole article we assume $p \nmid |A|$. If $C_A(V)$ denotes the kernel of the representation, we ask whether or not $\bar{A} := A/C_A(V)$ has a regular orbit as a permutation group on $V \setminus \{0\}$. That is, we look for an element $v \in V \setminus \{0\}$ such that $C_A(v) = C_A(V)$, where $C_A(v)$ denotes the centralizer in $A$ of $v \in V$. Elementary methods using Clifford-theory lead to a theorem concerning the direct product of relatively prime groups. As an application we prove theorems on nilpotent groups which were received with other methods by B. Bailey Hargraves in 1980 (*J. Algebra* **72** (1981)). 

## I. DEFINITIONS AND GENERAL METHODS

Here we give a notation of Kurzweil [3], whose ideas inspired most of the following:

DEFINITION (1.I). *Let p be a prime not dividing A.*

$$V_p(A) := \text{The class of all } \mathbb{F}_q A\text{-modules } V \text{ for arbitrary } p\text{-power } q.$$

$$S_p(A) := \{V \in V_p(A) \mid A \text{ has a regular orbit on } V\}.$$

$$\tau(A) = \{p \in \pi'(A) \mid V_p(A) = S_p(A)\}; \sigma(A) := \bigcap_{U \leqslant A} \tau(U).$$

It is easy to see that $\operatorname{Max}(\pi'(A) \setminus \sigma(A)) < |A| - 1$, indeed if $p \in \pi'(A)$ and $p > |A| - 1$ we have $m - 1 < |A| - 1 \leqslant q = p^s$, where $m$ denotes the number of minimal subgroups $B_i$ in $\bar{A} = A/C_A(V)$. Since $\dim_{\mathbb{F}_q} C_V(B_i) \leqslant \dim_{\mathbb{F}_q} V - 1 =: n - 1$ we get $|V \setminus \bigcup_{i=1}^m C_V(B_i)| \geqslant q^n - 1 - m(q^{n-1} - 1) =$



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$q^{n-1}(q-1)+m-1>0$, and there is a $v \in V$ which is not fixed by any minimal subgroup and therefore generates a regular orbit.

LEMMA (2.I). *If $V \in S_p(A)$ for every irreducible $\mathbb{F}_q A$-module $V$, every $\mathbb{F}_q A$-module is in $S_p(A)$.*

*Proof.* Any $V$ decomposes $V = \bigoplus_{i=1}^r V_i$ with $V_i$ irreducible; by assumption we have $v_i \in V_i$ with $C_A(v_i) = C_A(V_i)$, hence $C_A(\sum_i v_i) = \bigcap_{i=1}^r C_A(v_i) = \bigcap_{i=1}^r C_A(V_i) = C_A(V)$, the claim.

We can improve this lemma in the sense that is suffices to consider only absolutely irreducible modules. This enables us to use classical results about the dimensions, etc., of those modules which were originally stated for splitting-fields in characteristic 0.

LEMMA (3.I). *If every absolutely irreducible $\mathbb{F}_q A$-module $V$ over any appropriate finite field $\mathbb{F}_q$ is in $S_p(A)$ then $V_p(A) = S_p(A)$.*

*Proof.* Let $V \in V_p(A)$, $V$ irreducible over $\mathbb{F}_q$. Let $L$ be a finite splitting-field of $A$ with $L \geqslant \mathbb{F}_q$; then $V_L = L \otimes_{\mathfrak{F}_q} V = \bigoplus_{g \in G} W^g$, $G := \mathrm{Gal}(\mathbb{F}_q(\chi_W) : \mathbb{F}_q)$, where $\chi_W$ denotes the character of an absolutely irreducible constituent $W$ of $V_L$, which are all realizable over $\mathbb{F}_q(\chi_W)$. Let $w \in W$ be such that $C_A(w) = C_A(W)$ and $\tilde{w} = \sum_{g \in G} w^g \in V_{\mathbb{F}_q}$. If $\tilde{w}^a = \tilde{w}$ for $a \in A$ we have $w^a = w$, so $a \in C_A(W) = C_A(W^g)$ for all $g \in G$, hence $a \in C_A(V)$.

COROLLARY (4.I). *$\pi'(A) = \sigma(A)$ for all abelian groups $A$.*

*Proof.* Every absolutely irreducible $A$-module is linear, so $C_A(v) = C_A(V)$ for all $v \neq 0$.

If we use the fact that for two finite groups $A_1, A_2$ every absolutely irreducible $A_1 \times A_2$-module over the appropriate field $\mathbb{F}_q$ can be written in the form $V_1 \otimes_{\mathbb{F}_q} V_2$ with absolutely irreducible $\mathbb{F}_q A_i$-modules $V_i$, we get the main theorem of this paper:

THEOREM (5.I). *Let $A_1, A_2$ be finite groups with $(|A_1|, |A_2|) = 1$, $p \in \pi'(A_1) \cap \pi'(A_2)$ and let $V_p(A_i) = S_p(A_i)$ for $i = 1, 2$; then $V_p(A_1 \times A_2) = S_p(A_1 \times A_2)$.*

*Proof.* Let $V = V_1 \otimes_{\mathbb{F}_q} V_2$ be an absolutely irreducible $\mathbb{F}_q(A_1 \times A_2)$-module as described above. There exist $v \in V_1$, $w \in V_2$ with $C_{A_1}(v) = C_{A_1}(V_1)$, $C_{A_2}(w) = C_{A_2}(V_2)$. Let $B_1 = \{v = v_1, v_2, ..., v_n\}$, $B_2 = \{w = w_1, w_2, ..., w_m\}$ be $\mathbb{F}_q$-bases of $V_1$ and $V_2$, respectively. Then $C := \{v_i \otimes w_j \mid i = 1, 2, ..., n;\ j = 1, 2, ..., m\}$ is a basis of $V$. We claim that $C_{A_1 \times A_2}(v \otimes w) = C_{A_1 \times A_2}(V)$:

Let $(v \otimes w)(a_1, a_2) = v^{a_1} \otimes w^{a_2} = \sum_{i=1}^{n} \sum_{j=1}^{m} l_i \cdot k_j \cdot v_i \otimes w_j = v \otimes w$; then

$$l_i \cdot k_j = \begin{cases} 1 & i = j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence $l_1 \cdot k_1 = 1$ and $l_i = k_j = 0$ for all $i, j > 1$. Therefore $v^{a_1} = l_1 v$, $w^{a_2} = k_1 w$ and the $\mathbb{F}_q^*$-order $\operatorname{ord} k_1 \mid \operatorname{ord} |A_2|$, $\operatorname{ord} l_1 \mid \operatorname{ord} |A_1|$. Since $\operatorname{ord} k_1 = \operatorname{ord} l_1^{-1} = \operatorname{ord} l_1$ and $|A_1|$, $|A_2|$ are relatively prime we have $l_1 = k_1 = 1$ and $a_1 \in C_{A_1}(V_1)$, $a_2 \in C_{A_2}(V_2)$. Hence $(a_1, a_2)\ C_{A_1 \times A_2}(V)$. By applying (3.I) we get the statement of the theorem.

## II. $p$-Groups and Nilpotent Groups

Theorem (1.II) (Kurzweil [3]). *Let $P$ be a $p$-group, $p \neq 2$; then $\pi'(P) \backslash \sigma(P) \subseteq \{2\}$.*

*Proof.* Let us assume that the statement may be true for all $p$-groups of smaller order than $P$. Let $V \in V_{\bar{p}}(P) \backslash S_{\bar{p}}(P)$, $\bar{p} \neq 2$, $\bar{p} \neq p$, such that $P$ operates faithfully and irreducibly on $V$ over $\mathbb{F}_{\bar{p}^s}$. Since $P$ is not cyclic it contains an abelian noncyclic $N \lhd P$ and hence $V_N$ is not homogeneous. So $V = \bigoplus_{a \in [P:N_p(W)]} W^a$, where $W$ is a homogenous $N$-component with normalizer $N_p(W) \lneqq P$. Take a maximal $U < P$ such that $N \leqslant N_p(W) \leqslant U \lhd P$ and write $V = \bigoplus_{a_i \in [P:U]} W'^{a_i}$ with $a_0 = e$. Since $W' \in V_{\bar{p}}(U) = S_{\bar{p}}(U)$ there is $w \in W'$ such that $C_U(w) = C_U(W')$. Now choose $k \in \mathbb{F}_{\bar{p}^s}^*$ such that $\operatorname{ord} k = \bar{p} - 1$ and let $v := k \cdot w + w^{a_1} + \cdots + w^{a_{p-1}}$. We claim that $C_P(v) = e$: For this let $v^{u \cdot a_i} = v$; then $kw^{ua_i} \in W'^{a_i}$, hence for $i > 0$ we get $kw^{ua_i} = w^{a_i}$ and $w^u = 1/kw$. So $\operatorname{ord} k = \bar{p} - 1 \mid \operatorname{ord} u = p^x$. It follows that $\bar{p} - 1 = p^m$, contradicting $p$, $\bar{p} \neq 2$; so the only possibility is $i = 0$. That is, $v^u = v$, but then $w^u = w$. Since $w^{a_i u} = w^{u' a_i} \in W'^{a_i}$, we get $w^{a_i u} = w^{a_i}$ for all $i$. So $u \in \bigcap_i C_U(w^{a_i}) = \bigcap_i C_U(W'^{a_i}) = C_U(V) = e$. We finally get $V \in S_{\bar{p}}(P)$, a contradiction.

A trivial example for an odd $p$-group with $\pi'(P) \backslash \sigma(P) = \{2\}$ is the following: Let $A \in \operatorname{Syl}_3(\operatorname{Gl}_6(\mathbb{F}_2))$, i.e., a 3-sylowgroup of the set of nonsingular $6 \times 6$-matrices. It is easy to see that $A \cong \mathbb{Z}_3 \sim \mathbb{Z}_3$, the wreath-product. $|A| = 3^4 > 63 = |(\mathbb{F}_2)^6 \backslash \{0\}|$. So obviously $A$ cannot have a regular orbit on $(\mathbb{F}_2)^6$.

Theorem (2.II). *Let $G$ be a 2-group then $\pi'(G) \backslash \sigma(G) \subseteq \hat{M} \cup \hat{F}$, where $\hat{M}$ resp. $\hat{F}$ denote the Mersenne resp. the Fermat primes.*

*Proof.* Let $G$ be minimal with $2 \neq p \in \tau(G)$ and $V \in V_p(G) \backslash S_p(G)$ such that $G$ operates faithfully and absolutely irreducible on $V$. If $G$ contains a noncyclic abelian normal subgroup $N$, then we proceed as in the odd case and get the analogous equation: $p-1=2^m$, wherefore $p=2^m+1$ is a Fermat-prime. Otherwise it is well known that $G$ contains a maximal cyclic normal subgroup $N$ with $[G:N]=2$. (See Huppert [1, p. 304]). Since $V$ is absolutely irreducible, we have dim $V=2$ and $N$ operates fixed-point freely on $V\backslash\{0\}$. From our assumption we have $v^N=v^G$ for all $v\in V\backslash\{0\}$, so $|C_G(v)|=2$ for all $v\neq 0$ and $C_G(v)\cap N=e$. Let $L_1(V)$ denote the class of all 1-dimensional subspaces of $V$ and $I$ the class of all involutions in $G$. Consider

$$\theta\colon L_1(V)\to I\backslash N,\ \mathbb{F}_q v\mapsto C_G(v)=\langle t_v\rangle$$
$$\psi\colon I\backslash N\to L_1(V),\ t\mapsto C_V(t).$$

We claim that these maps are bijections. Obviously it suffices to prove the surjectivity of $\theta$: Let $h$ be the involution in $N$. Then $C_V(h)=0$, hence $v^h+v=0$ for all $v\in V$. Let $t\in I\backslash N$; then there must be $w\in V$ with $-w^t\neq w$, because $t\neq h$. So $0\neq w^t+w\in C_V(t)$, which must therefore have dimension 1.

It follows that $|I\backslash N|=|L_1(V)|=p^{2s}-1/p^s-1=p^s+1$; $|I|=p^s+2$. There are only three semidirect products $N\cdot\langle t\rangle$: the semidihedral $\mathrm{SD}_{2^n}$ with $|I|=2^{n-2}+1$, the dihedral $\mathrm{D}_{2^n}$ with $|I|=2^{n-1}+1$, and another group with three involutions, which cannot be our minimal counterexample (see Huppert [1, p. 91]).

We conclude $p^s=2^{n-2}-1$ or $p^s=2^{n-1}-1$ and hence $s=1$, $p=2^m-1$, a Mersenne prime.

The next theorem shows that the exceptional cases of (2.II) really exist in a natural situation:

THEOREM (3.II). *Let $p>2$ and $G\in \mathrm{Syl}_2(Gl_2(\mathbb{F}_p))$ then the following are equivalent*:

(i) *$G$ does not have a regular orbit on $(\mathbb{F}_p)^2$;*

(ii) *$p$ is a Mersenne prime or $p$ is a Fermat prime $>3$,*

*Proof.* (i) $\Rightarrow$ (ii) Follows from (2.II).

(ii) $\Rightarrow$ (i) Let $p=2^r-1$ be a Mersenne prime. Since $4\nmid p-1$ $G$ is semidihedral. $|Gl_2(\mathbb{F}_p)|=(p^2-1)(p^2-p)=p(p-1)^2(p+1)=p\cdot k^2\cdot 2^{r+2}$, $k=1 \bmod(2)$, so $|G|=2^{r+2}\cdot|SL_2(\mathbb{F}_p)|=p(p-1)(p+1)=pk2^{r+1}$. Let $S\in \mathrm{Syl}_2(SL_2(\mathbb{F}_p))$, $|S|=2^{r+1}$. Choose $G$ such that we have $S\lhd G$, $[G:S]=2$. For all $g\in G$, $k$, $z\in\mathbb{F}_p\backslash\{0\}$, $\binom{k}{0}^g=\binom{z}{0}$ implies $k\equiv \pm z \bmod p$ (**) because $z/k$ is an eigenvalue of $g$ and so ord $z/k$ in $\mathbb{F}_p^*$ divides ord $g=2^m$ and $p-1$, wherefore it has order 1 or 2. $S$ operates regularly on $\mathbb{F}_p^2\backslash\{0\}$, for otherwise we would have $v^s=v\neq 0$ for some $s\in S\backslash\{e\}$. With respect to a basis $\{v, w\}$ $s$ would have the form $\left(\begin{smallmatrix}1 & a\\ 0 & b\end{smallmatrix}\right)$ with $b=1$ since $\det s=1$, so $s \triangleq \left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)$ and $\operatorname{ord} s=p$, contradicting $s \in S$.

$\mathbb{F}_p^2$ splits under $S$ in $p^2-1/2^{r+1}=(p+1)(p-1)/2^{r+1}=(p-1)/2$ regular orbits. Since $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right) \in S$, these are:

$$\begin{array}{ccccccc}\begin{pmatrix}1\\0\end{pmatrix}^S & \neq & \begin{pmatrix}2\\0\end{pmatrix}^S & \neq & \cdots & \neq & \begin{pmatrix}p-1/2\\0\end{pmatrix}^S\\ \uplus & & \uplus & & & & \uplus\\ \begin{pmatrix}-1\\0\end{pmatrix} & & \begin{pmatrix}-2\\0\end{pmatrix} & & & & \begin{pmatrix}p+1/2\\0\end{pmatrix}\end{array}.$$

A regular $G$-orbit would be a union of two $S$-orbits and so would contradict (**). Therefore every $v \in \mathbb{F}_p^2$, $v \neq 0$ is centralized by an involution in $G$, lying in the maximal dihedral-subgroup of $G$, which does not have a regular orbit either. If $p=2^{2^t}+1>3$, $|G|=2(2^{2^t})^2=2\cdot 2^{2^t\cdot 2}=2^{2^t 2+1}=2^{2^{t+1}+1}$, whereas

$$\begin{aligned}|\mathbb{F}_p^2\backslash\{0\}| &= (p-1)(p+1)=2^{2^t}\cdot(2^{2^t}+2)=2^{2^t\cdot 2}+2^{2^t+1}\\ &= 2^{2^t+2^t}+2^{2^t+1}<2\cdot 2^{2\cdot 2^t}=|G|.\end{aligned}$$

So $G$ has no regular orbit on $\mathbb{F}_p^2$.

This together with (5.I) gives the following result:

THEOREM (4.II). *Let G be a nilpotent group, then the following holds*:

(a) $|G|=1 \bmod (2) \Rightarrow \pi'(G)\backslash\sigma(G) \subseteq \{2\}$;

(b) $|G|=0 \bmod (2) \Rightarrow \pi'(G)\backslash\sigma(G) \subseteq \hat{M} \cup \hat{F}$;

(c) $p \in \pi'(G)$, $p=2^r-1 \in \hat{M}$, $p>3$:

$p \in \sigma(G) \Leftrightarrow$ *the* 2-*sylowsubgroups of* $G$ *are* $D_{2^{r+1}}$- and $SD_{2^{r+2}}$-*free.*

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